## Statistics and learning

An introduction: from data to modelling

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## Statistical approach

A quick, partial and not very comprehensive overview

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- Grail: linking data to mathematical modelling, objectivelly quantify and interpret conclusions and...awareness of limitations: statistics helps but won't make decision for you !


## Inspiring work / our bibliography

T. Hastie, R. Tibshirani and J. Friedman. Elements of statistical learning.
Springer, 2nd edition, 2009.
E. Moulines, F. Roueff and J.-L. Pac (and formerly F. Rossi)
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S. Lousteau

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http://www.math.univ-angers.fr/ loustau/, 2013.

And many others we just forgot to mention.

## From data to modelling

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We can denote by $x_{i}$ successive opinions taking (binary) values " A " $(=0)$ or "B" (=1). Mathematician sees that as realisation of random variables denoted $X_{i}$.

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...arises from the choice of the questioned persons, NOT from in each actual answer.

Incidental reminder: Bernouilli distribution, with parameter $0<p<1$...

- laid question: is $p_{0}>1 / 2$ or $<1 / 2$ ? This is a test.


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1. describe sample using estimates
2. quantitatively answer the question (generalising sample to full population conclusions)

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a mandatory step

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- remember the Jean Tibéri vs. Lyne Cohen-Solal (+ Ph. Meyer) council election in Paris in 2008 between 20.45 and 21.15 ? At 20.45, ( $463 ; 409 ; 106$ ) but after counting the votes : (11, 044; 11, 269; 2, 730).


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- Construction of confidence intervals to answer the question.


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- (almost never use skewness and kurtosis)


## Two important probabilistic tools in statistics

Law of large numbers

Theorem
Let $X_{1} \ldots X_{n}$ be iid random variables with mean $\mu$. Then the empirical mean converges in probability towards $\mu$, i.e.:

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In other term, for all $\epsilon>0, P\left(\left|\overline{X_{n}}-\mu\right|>\epsilon\right) \rightarrow 0$

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Let $X_{1} \ldots X_{n}$ be iid random variables which admit an order 2 moment. Denote by $\mu$ and $\sigma$ the corresponding mean and standard deviation, then:

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In the case of distribution with density functions, this means that

$$
P\left(\frac{\sqrt{n}}{\sigma}\left(\overline{X_{n}}-\mu\right) \leq x\right):=F_{n}(x) \longrightarrow P(Z \leq x)=\frac{\int_{-\infty}^{x} \mathrm{e}^{-z^{2} / 2} d z}{\sqrt{2 \pi}}
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- Our intuition and the LLN tell us that $p_{0}$ is "close" to 0.42 .
- Can we conclude ? Is this estimate enough ?

Let's play around the Central limit theorem...

## Concluding the example

at the price of a slight risk

- we directly derive $\frac{\sqrt{n}}{\sqrt{p_{0}\left(1-p_{0}\right)}}\left(\overline{X_{n}}-p_{0}\right) \rightarrow \mathcal{N}(0,1)$ so


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- is the conclusion similar if $n=1,000$ ?

Note: $95 \%$ could have been replaced by $99 \%$. How could this have affected the conclusion ? What about $100 \%$ ?

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- lessons from this: tests are not reductible to confidence intervals and...don't be fooled by an obscure choice of hypotheses !


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